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AMPLIFICATION OF THE TRAVELING WAVE TUBE

by

Bernard Friedman

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Abstract

This paper presents a method for solving the matching equation given by Chu and Jackson "Field Theory of Traveling Wave Tubes", (Proc. I.R.E., Vol. 36, July, 1948, p. 853). The solution of this equation gives modes of the helix with the electron beam and thus determines the amplification factor. The essential idea is to consider the modes as a perturbation of the modes produced by the cold helix. The dependence of the amplification factor on the parameters of the tube can be explicitly determined. One surprising result is that the tube will not amplify for too large values of the D.C. electron beam.

Introduction

In a recent paper, Chu and Jackson[1] have investigated the modes of an idealized traveling wave tube. They show that the propagation constant γ_n associated with the nth mode satisfies a complicated transcendental equation. By making some rough approximations, the equation reduces to a cubic whose solutions can be found. However, their method is subject to the criticism that in order to perform the reduction, all the parameters of the problem must be fixed. This means that when any of the parameters is changed, the whole procedure for solving the equation must be repeated. As a result, the effect upon the propagation constants of a change in parameters can not be determined.

In this report a simpler and more exact method is presented for solving the transcendental equation given by Chu and Jackson. The method yields explicitly the manner in which the propagation constants depend on the parameters so that the effect of a change of parameters can be readily calculated. In particular the maximum amplification and the band width of the tube may be found.

The theory and method presented here have been applied to some cases of the traveling wave tube. It has been difficult to compare the theoretical results with the experimental ones because the radius of the electron beam is very difficult to measure. It is hoped that this difficulty may be overcome so that an adequate comparison with experiment could be made.

In section 1, we have given a summary of that part of the paper of Chu and Jackson which yields the basic matching equation.

In section 2, the basic equation is reduced to a cubic equation. The coefficients of the cubic equation are explicitly given in terms of the geometry of the tube and the ratio $\frac{I}{V^{3/2}}$ of the electron beam. (I is the d.c. current and V is the voltage of the electron beam).

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[1] Chu and Jackson, "Field Theory of Traveling Wave Tubes", Proc.I.R.E. (Vol. 36, July 1948 p. 853).

In section 3, the question of maximum amplification is investigated. A method for finding maximum amplification is given. From the theory of this section, graphs are given for maximum amplification in terms of the ratio $\frac{I}{V^{3/2}}$. Graphs have also been constructed for the quantity d_o in terms of $\frac{I}{V^{3/2}}$, where d_o is the difference between the wave number for the electron beam and the wave number of the "cold" helix.

In section 4, an investigation is made of the case where ak is large. Asymptotic formulas for the Bessel functions are used and the cubic equation of section 2 is reduced to a simpler cubic. The amplification of the wave, in this case, can be explicitly written down in terms of the parameters of the tube. Also it is shown that the tube will not amplify for too large values of the d.c. current.

1. Summary of the principal result of Chu and Jackson.

In the paper by Chu and Jackson the actual helix is replaced by a lossless helical sheath of radius a and an infinitesimal thickness. The current flow along the sheath is constrained to move in a direction which makes a constant angle $90^\circ - \theta$ with the axis of the helix. The force acting on the electrons is restricted to that associated with the longitudinal electric field only; and the electrons are assumed to have no initial transverse motion. It is also assumed that the electrons are confined within a cylinder of radius b concentric with the helical sheath. The time-average beam current density is assumed constant over the cross section.

Chu and Jackson tackle the problem as a boundary value problem. Three regions are defined: the region outside the cylinder, the region between the cylinder and the stream of electrons and finally the region occupied by the electrons, that is, $r < b$. Field components for E and H are deduced from Maxwell's Equations and the fields are matched on the two boundaries. If the fields are assumed to be circularly symmetric

and vary with $e^{i\omega t - \gamma z}$, the transcendental equation for the propagation constant γ is found to be

(1.1)
$$i\,\frac{ka}{pa}\,\frac{\eta}{p}\,\frac{I_1(\eta b)}{I_0(\eta b)} = i\,\frac{ka}{pa}\left[\frac{I_1(pb)-K_1(pb)\left(\dfrac{\left(\frac{ka}{pa}\right)^2\cot^2\theta I_1(pa)K_1(pa)-I_0(pa)K_0(pa)}{K_0^2(pa)}\right)}{I_0(pb)+K_0(pb)\left(\dfrac{\left(\frac{ka}{pa}\right)^2\cot^2\theta I_1(pa)K_1(pa)-I_0(pa)K_0(pa)}{K_0^2(pa)}\right)}\right]^{*}$$

where $I_0(x)$, $I_1(x)$, $K_0(x)$ and $K_1(x)$ are modified Bessel functions: and where

(1.2) $$k^2 = \omega^2 \varepsilon \mu, \quad k = \frac{2\pi}{\lambda} = \frac{\omega}{c}$$

(1.3) $$p^2 = -(\gamma^2 + k^2)$$

(1.4) $$\eta^2 = p^2\left[1 + \frac{\frac{e}{m} I}{\pi b^2 \varepsilon v_0^3 \left(i\frac{\omega}{v_0} - \gamma\right)^2}\right]$$

(1.5) $\frac{e}{m}$ = ratio of charge of electron to mass

(1.6) v_0 = average value of the electron beam velocity.

(1.7) I = d.c. beam current

In order to discuss the behavior of the tube, Eq. (1.1) must be solved for p. A real root of this equation corresponds to a progressive wave while a complex root with a positive imaginary part indicates that the corresponding wave will be amplified. Chu and Jackson indicate a method for its solution. They plot the right hand side of Eq. (1.1) for a particular choice of parameters. Its graph suggests that it can be replaced by the function

(1.8) $$\text{R.H.S.} = ika\,C\,\frac{p - p'}{p - p''} .$$

The left hand side of Eq. (1.1) is expanded into a power series in $\frac{\eta}{p}$ and the first term

(1.9) $$\text{L.H.S.} = i\,\frac{ka}{a}\,\frac{\eta^2}{2}\,\frac{b}{2}$$

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* See Chu and Jackson "Field Theory of Traveling Wave Tubes", Proc. I.R.E. Vo. 36, July 1948, pp. 853-863, Equations (20), (25).

only is retained. Using Eqs. (1.4),(1.8) and (1.9) there results the equation

$$(1.10) \qquad \frac{b}{a} \quad 1 + \frac{\frac{e}{m} I}{\pi b^2 \varepsilon v_0^3 (i \frac{\omega}{v_0} - \gamma)^2} = 2C \left(\frac{\gamma - i\beta'}{\gamma - i\beta''}\right),$$

where $p \cong - i\gamma$, $p' = i\beta'$, $p'' = i\beta''$. The equation $p \cong - i\gamma$ follows from Eq.(1.3) and the observation that the phase velocities of the waves are much less than that of light. Eq.(1.10) is the equation derived by Chu and Jackson. Its derivation, as has been stated in the introduction, depends on fixing all the parameters of the equation (1.1.)

2. Approximate Solution of the Fundamental Equation.

In this section a method will be given for solving Eq. (1.1). Rewrite Eq.(1.1) in the form

$$(2.1) \qquad \frac{\eta}{p} \frac{I_1(\eta b)}{I_0(\eta b)} \cdot \frac{I_0(pb)}{I_1(pb)} = \frac{1 - \frac{K_1(pb)}{I_1(pb)} G(pa)}{1 + \frac{K_0(pb)}{I_0(pb)} G(pa)}$$

where

$$(2.2) \quad G(pa) = \frac{I_0(pa)}{K_0(pa)} \left[\left(\frac{ka \cos \theta}{pa}\right)^2 \frac{I_1(pa)}{I_0(pa)} \frac{K_1(pa)}{K_0(pa)} - 1 \right] .$$

In the absence of an electron beam the modes are given by the equation

$$(2.3) \qquad G(pa) = 0.$$

The crux of the present method is to consider the roots of Eq. (2.1) as being found by means of a perturbation of the roots of Eq. (2.3) .

Let p_0a be the single root of Eq.(2.3). Put

$$(2.4) \qquad p_0 a = x_0 \quad , \quad pa = x_0 + y$$

where y is assumed to be small compared to x_0. The right hand side of Eq. (2.1) may be written as

$$(2.5)\quad \text{R.H.S.} = 1 - \frac{G(pa)\left\{\dfrac{K_1(pb)\,I_0(pb) + K_0(pb)\,I_1(pb)}{I_1(pb)\,I_0(pb)}\right\}}{1 + \dfrac{K_0(pb)}{I_0(pb)}\,G(pa)}$$

$$= 1 - \frac{\dfrac{G(pa)}{pb\,I_1(pb)\,I_0(pb)}}{1 + \dfrac{K_0(pb)}{I_0(pb)}\,G(pa)} \quad ,$$

where use has been made of the Wronskian relationship

$$(2.6)\quad K_1(pb)\,I_0(pb) + K_0(pb)\,I_1(pb) = \frac{1}{pb} \quad .$$

If now G(pa) is expanded in a Taylor's series about x_0, there results

$$(2.7)\quad G(pa) = G(x_0 + y) = G(x_0) + y\,G'(x_0) + \ldots$$

$$= y\,G'(x_0) + \ldots$$

$$= y\,G'(x_0), \text{ approximately.}$$

Replacing G(pa) in Eq. (2.5) by its value, it is found that

$$(2.8)\quad \text{R.H.S.} = 1 - \frac{\dfrac{y\,G'(x_0)}{\frac{b}{a}\,x_0 I_0(\frac{b}{a}\,x_0)\,I_1(\frac{b}{a}\,x_0)}}{1 + \dfrac{K_0(\frac{b}{a}\,x_0)}{I_0(\frac{b}{a}\,x_0)}\,G'(x_0)y} \quad .$$

For the left hand side of the fundamental equation we proceed as follows. Expand the function $\frac{I_1(rx)}{I_o(rx)}$ in a Taylor's series in r about r = 1. We find that

$$(2.9)\quad \frac{I_1(rx)}{I_o(rx)} = \frac{I_1(x)}{I_o(x)} + (r-1)\,x\left[\frac{I_1'(x)\,I_o(x) - I_1(x)\,I_o'(x)}{I_o^2(x)}\right] + \dots$$

$$= \frac{I_1(x)}{I_o(x)} + (r-1)x\left[\frac{I_o^2(x) - \frac{1}{x}\,I_o'(x)\,I_o(x) - I_1^2(x)}{I_o^2(x)}\right].$$

From Eq. (2.9) it follows that

$$\frac{I_1(rx)}{I_o(rx)}\,\frac{I_o(x)}{I_1(x)} = 1 + (r-1)x\left[\frac{I_o(x)}{I_1(x)} - \frac{I_1(x)}{I_o(x)} - \frac{1}{x}\right]$$

If we let

$$r = \frac{\eta}{p},$$

the left hand side of the fundamental equation is

$$\text{L.H.S.} = \frac{\eta}{p} + \left(\frac{\eta}{p} - 1\right)\frac{\eta}{p}\left[\frac{I_o(pb)}{I_1(pb)} - \frac{I_1(pb)}{I_o(pb)} - \frac{1}{pb}\right] pb$$

$$= \frac{\eta}{p} + \left(\frac{\eta}{p} - 1\right)\frac{\eta}{p}\left[\frac{I_o(\frac{b}{a}x_o)}{I_1(\frac{b}{a}x_o)} - \frac{I_1(\frac{b}{a}x_o)}{I_o(\frac{b}{a}x_o)} - \frac{1}{\frac{b}{a}x_o}\right]\frac{b}{a}x_o$$

$$= \frac{\eta}{p} + \left(\frac{\eta}{p} - 1\right)\frac{\eta}{p}\,\delta,$$

where

$$(2.11)\quad \delta = \left[\frac{I_o(\frac{b}{a}x_o)}{I_1(\frac{b}{a}x_o)} - \frac{I_1(\frac{b}{a}x_o)}{I_o(\frac{b}{a}x_o)} - \frac{1}{\frac{b}{a}x_o}\right]\frac{b}{a}\,x_o$$

Assuming $\frac{\eta}{p} = 1$, the Eq. (2.11) becomes

$$\text{L.H.S.} = \frac{\eta}{p} + \left(\frac{\eta}{p} - 1\right)\delta \,. \tag{2.12}$$

Equating Eq. (2.8) and Eq. (2.12) the basic equation is replaced by

$$\frac{\eta}{p} + \left(\frac{\eta}{p} - 1\right)\delta = 1 - \frac{\dfrac{y\,G'(x_o)}{\frac{b}{a}x_o\, I_o(\frac{b}{a}x_o)\, I_1(\frac{b}{a}x_o)}}{1 + \dfrac{K_o(\frac{b}{a}x_o)}{I_o(\frac{b}{a}x_o)}\,G'(x_o)y} \,, \tag{2.13}$$

or

$$\left(\frac{\eta}{p} - 1\right)(1 + \delta) = - \frac{\dfrac{y\,G'(x_o)}{\frac{b}{a}x_o\, I_o(\frac{b}{a}x_o)\, I_1(\frac{b}{a}x_o)}}{1 + \dfrac{K_o(\frac{b}{a}x_o)}{I_o(\frac{b}{a}x_o)}\,G'(x_o)} \,. \tag{2.14}$$

We know, however, that

$$\frac{\eta^2}{p^2} = 1 + \frac{\frac{e}{m}I}{\pi b^2 \varepsilon v_o^3 (i\frac{\omega}{v_o} - \gamma)^2} \tag{1.4}$$

But the phase velocities of the beam are much less than that of light, thus (see Eq. (1.3))

$$\gamma = i\,p \,. \tag{2.15}$$

Hence Eq. (1.4) is

$$\frac{\eta^2}{p^2} = 1 - \frac{\frac{e}{m} I}{\pi b^2 \varepsilon v_o^3 \left(\frac{\omega}{v_o} - p\right)^2}$$

(2.16)
$$= 1 - \frac{\frac{e}{m} I}{\pi \frac{b^2}{a^2} \varepsilon v_o^3 \left(\frac{\omega}{v_o} a - x_o + x_o - ap\right)^2}$$

$$= 1 - \frac{\frac{e}{m} I}{\pi \frac{b^2}{a^2} \varepsilon v_o^3 (d_o - y)^2} ,$$

where

(2.17)
$$d_o = \frac{\omega}{v_o} a - x_o .$$

Hence

$$\frac{\eta}{p} = 1 - \frac{\frac{e}{m} I}{2\pi \frac{b^2}{a^2} \varepsilon v_o^3 (d_o - y)^2}$$

(2.18)

$$= 1 - \frac{1}{\pi \varepsilon 4\sqrt{2} \left(\frac{e}{m}\right)^{1/2} \frac{b^2}{a^2}} \frac{I}{V^{3/2}} \cdot \frac{1}{(d_o - y)^2}$$

$$= 1 - \frac{W}{b^2/a^2} \frac{1}{(d_o - y)^2}$$

where

$$W = \frac{\frac{e}{m}}{\pi \frac{b^2}{a^2}} \; \frac{I}{v_0^3} = \frac{1}{\pi \varepsilon \, 2\sqrt{2} \, (\frac{e}{m})^{1/2} \, \frac{b^2}{a^2}} \; \frac{I}{V^{3/2}}$$

since $1/2 \; m \; v_0^2 = eV$. Combining Eqs. (2.18), (2.14) and (2.8), we get

$$y(y-d_0)^2 - \zeta(1 + \xi y) = 0 \tag{2.20}$$

where

$$\zeta = \frac{x_0 I_0(\frac{b}{a} x_0) \, I_1(\frac{b}{a} x_0)(1 + \delta) \, W}{2 \frac{b}{a} \; G'(x_0)} \tag{2.21}$$

$$\xi = \frac{K_0(\frac{b}{a} x_0)}{I_0(\frac{b}{a} x_0)} \; G'(x_0) \; . \tag{2.22}$$

Equation (2.20) is the desired cubic equation, ζ and ξ are known quantities depending on the geometry of the tube, and the voltage and current of the electron beam. In the case where the electron beam travels at such a speed that it is in exact step with the wave traveling along the helix, we have $d_0 = 0$. Thus d_0 may be considered a measure of how far from resonance the tube is.

The cubic equation (2.20) has real coefficients. It has three real roots, or one real root and a pair of complex roots. The case of three real roots (γ pure imaginary) corresponds to modes which are neither attenuated nor amplified. In the other case, however, the pair of complex conjugate roots give rise on the one hand to an attenuated wave, on the other hand to an amplified wave. The amount of attenuation and amplification depends on the imaginary part of the complex roots of Eq. (2.20). We shall investigate this question in the next section.

3. Maximum Amplification of the Tube.

The question of what values of the parameters give maximum amplification will now be taken up. It will be shown that maximum amplification occurs for a value of $d_0 > 0$ so that the velocity of the electron beam should be slightly less than that of the wave down the tube. But the maximum is so broad that $d_0 = 0$ will give amplification close enough to maximum.

We start with our basic equation

$$(3.1) \qquad y(y-d_0)^2 - \zeta(1 + \xi y) = 0$$

and pose the problem of determining the value of d_0 which will give maximum amplification.

Let

$$(3.2) \qquad y = z\,\zeta^{1/3}, \quad \tau = d_0/\zeta^{1/3}, \quad \rho = \xi\,\zeta^{1/3}$$

then Eq. (3.1) becomes

$$(3.3) \qquad z(z-\tau)^2 - (1 + \rho z) = 0,$$

and we ask what value of τ will yield the maximum imaginary part for a complex root of Eq. (3.3).

Let

$$(3.4) \qquad z = \alpha + i\beta,$$

then, since all coefficients are real, the imaginary part of Eq. (3.3) may be written as

$$(3.5) \qquad \beta^2 = 3\alpha^2 - 4\alpha\tau + \tau^2 - \rho.$$

Let z_0 be the real root of Eq. (3.3), then

$$(3.6) \qquad z_0 + 2\alpha = 2\tau, \quad \alpha = \tau - \frac{z_0}{2}$$

Differentiating Eq. (3.5) with respect to τ yields

$$(3.7)\quad \frac{d\beta^2}{d\tau} = b\alpha \frac{d\alpha}{d\tau} - 4\tau \frac{d\alpha}{d\tau} - 4\alpha + 2\tau = 0$$

for a maximum, or

$$(3.8)\quad \frac{d\alpha}{d\tau} = \frac{2\alpha - \tau}{3\alpha - 2\tau} = \frac{\tau - z_0}{\tau - \frac{3}{2} z_0} .$$

But

$$(3.9)\quad \frac{d\alpha}{d\tau} = 1 - \frac{1}{2} \frac{dz_0}{d\tau} ,$$

and

$$(3.10)\quad 3 z_0^2 \frac{dz_0}{d\tau} - 4\tau z_0 \frac{dz_0}{d\tau} + (\tau^2 - \rho) \frac{dz_0}{d\tau} - 2z_0^2 + 2\tau z_0 = 0$$

so that

$$(3.11)\quad \frac{dz_0}{d\tau} = \frac{2z_0^2 - 2\tau z_0}{3z_0^2 - 4\tau z_0 + \tau^2 - \rho} = 2\left(1 - \frac{d\alpha}{d\tau}\right) = - \frac{z_0}{\tau - \frac{3}{2} z_0} .$$

Eq. (3.11) reduces to

$$2\tau z_0 = 2\tau^2 + 2\rho ,$$

or

$$(3.12)\quad z_0 = \tau + \frac{\rho}{\tau} .$$

Using the value of z_0 given by Eq. (3.12) in Eq. (3.3) gives us an equation which reduces to

$$(3.13)\quad \tau^3 (1 + \rho\tau) = \rho^3 .$$

Eq. (3.13) is the solution to our problem. A graph of τ as a function of ρ is given in Figure 1.

Now use Eq. (3.12) in Eq. (3.5) to get

$$\beta^2 = \frac{1}{4}\left(\frac{3\rho}{\tau} + \tau\right)\left(\frac{\rho}{\tau} - \tau\right) . \tag{3.14}$$

Eq. (3.14) provides the maximum imaginary part for a given τ.

Finally, in terms of the original parameters, Eq. (3.13) becomes

$$d_o^3\,(1 + \xi d_o) = \xi^2 \zeta^2 , \tag{3.15}$$

and since the imaginary part of y is $\beta \zeta^{1/3}$, Eq. (3.14) becomes

$$\beta^2\ \zeta^{2/3} = \frac{1}{4}\left(\frac{3\xi\zeta}{d_o} + d_o\right)\left(\frac{\xi\zeta}{d_o} - d_o\right) .$$

4. Discussion for large values of ka.

Further discussion of the problem is possible if it is assumed that ak is large, say about 3 or greater. The Eq. (1.1) again reduces to a cubic equation but since the Bessel functions may, in this case, be replaced by their asymptotic values, the coefficients ξ and ζ of Eq. (2.20) are considerably simplified.

First, we note that Eq. (2.3) becomes

$$\left(\frac{pa}{ka}\right)^2 \tan^2 \theta = 1, \tag{4.1}$$

so that

$$x_o = ka \cot \theta \tag{4.2}$$

Assume that the electron beam is in step with the wave, that is

$$d = 0 = \frac{\omega a}{v_o} - x_o = \frac{\omega a}{v_o} - ak \cot \theta \tag{4.3}$$

so that

$$v_o = c \tan \theta . \tag{4.4}$$

Eq. (2.20) reduces to

$$y^3 - \zeta\xi y - \zeta = 0 \,, \tag{4.5}$$

where for the sake of convenience we shall rewrite ζ and ξ. We have

$$\zeta = \frac{x_o \, I_o(\frac{b}{a} x_o) \, I_1(\frac{b}{a} x_o)(1+\delta) \, W}{2 \, \frac{b}{a} \quad G'(x_o)} \tag{4.6}$$

$$\xi = \frac{K_o(\frac{b}{a} x_o)}{I_o(\frac{b}{a} x_o)} \; G'(x_o) \;, \tag{4.7}$$

where

$$W = \frac{1}{2\sqrt{2}\sqrt{\frac{e}{m}} \; \frac{b^2}{a^2}} \cdot \frac{I}{V^{3/2}} = \ell \, \frac{I}{V^{3/2}} \quad \text{say}, \tag{4.8}$$

and

$$(1+\delta) = \frac{b}{a} \; x_o \left[\frac{I_o(\frac{b}{a} x_o)}{I_1(\frac{b}{a} x_o)} - \frac{I_1(\frac{b}{a} x_o)}{I_o(\frac{b}{a} x_o)} \right] . \tag{4.9}$$

If use is made of the asymptotic formulas

$$I_\nu(x) \cong \frac{1}{\sqrt{2\pi x}} \, e^{x} \;, \quad K_\nu(x) \cong \sqrt{\frac{\pi}{2x}} \, e^{-x} \;, \; \nu = 0,1, \tag{4.10}$$

the coefficients ζ, $\zeta\xi$ of Eq. (4.5) reduce to

$$\zeta = - \frac{(\frac{a}{b})^4 x_o \;\; \ell}{8 \, e^{-2x_o(\frac{b}{a} - 1)}} \; \frac{I}{V^{3/2}} \;, \tag{4.11}$$

$$\zeta\xi = \frac{(\frac{a}{b})^4 \, \ell}{4} \; \frac{I}{V^{3/2}} \;. \tag{4.12}$$

Hence Eq. (4.5) becomes

$$Ay^3 - LA\,y + L = 0, \tag{4.13}$$

where

$$L = \left(\frac{a}{b}\right)^2 \frac{\ell}{4} \frac{I}{V^{3/2}} \tag{4.14}$$

$$A = \frac{2}{x_o} e^{-2x_o\left(\frac{b}{a} - 1\right)} \quad . \tag{4.15}$$

Put

$$y = \left(\frac{L}{A}\right)^{1/3} u \; , \tag{4.16}$$

then Eq. (4.13) becomes

$$u^3 - B\,u + 1 = 0, \tag{4.17}$$

where

$$B = A^{2/3} L^{1/3} = (A^2 L)^{1/3} \quad . \tag{4.18}$$

Eq. (4.17) will have imaginary roots if

$$B < \frac{3}{\sqrt[3]{4}} \quad . \tag{4.19}$$

The quantity B, however, contains the term $\left(\frac{I}{V^{3/2}}\right)^{1/3}$. Thus if all other quantities are kept constant, we see that no amplification occurs if the above ratio becomes too large.

Another consequence of Eq. (4.17) is as follows: Suppose

$$B \ll 1 \; , \tag{4.20}$$

This implies that u is approximately 1. A better approximation is

$$(4.21) \qquad u = -\sqrt[3]{1 + B} = -1 - \frac{B}{3} + \ldots$$

$$= -1 - \frac{B}{3} .$$

But if the real root of Eq. (4.17) is given by Eq. (4.21), the quadratic equation yielding the imaginary roots is given by

$$(4.22) \qquad u^2 - u(1 + \frac{B}{3}) + (1 - \frac{B}{3}) = 0,$$

or

$$(4.23) \qquad u = \frac{1 + \frac{B}{3} \pm \sqrt{-3 + 2B}}{2}$$

Hence the imaginary part is

$$(4.24) \qquad \text{Im} u = \pm \frac{\sqrt{3-2B}}{2} = \pm \frac{\sqrt{3}}{2} (1 - \frac{2}{3} B) .$$

An example of the use of results of this section will be given. Let $x_0 = ka \cot \theta = 2.4$; $\frac{b}{a} = .8$, $I = 25$ ma, $V = 2500$. Then Eq. (4.14), (4.15) become

$$L = 2.35 \times 10^{-3} , \quad A = 2.18 .$$

Thus

$$(\frac{L}{A})^{1/3} = 1.024 \times 10^{-1} , \quad B = (A^2 L)^{1/3} \equiv 2.2 \times 10^{-1} ,$$

and $\frac{1}{2} \sqrt{3-2B} = .80$. Hence Eq. (4.16) gives for the imaginary part of the root of Eq. (4.13)

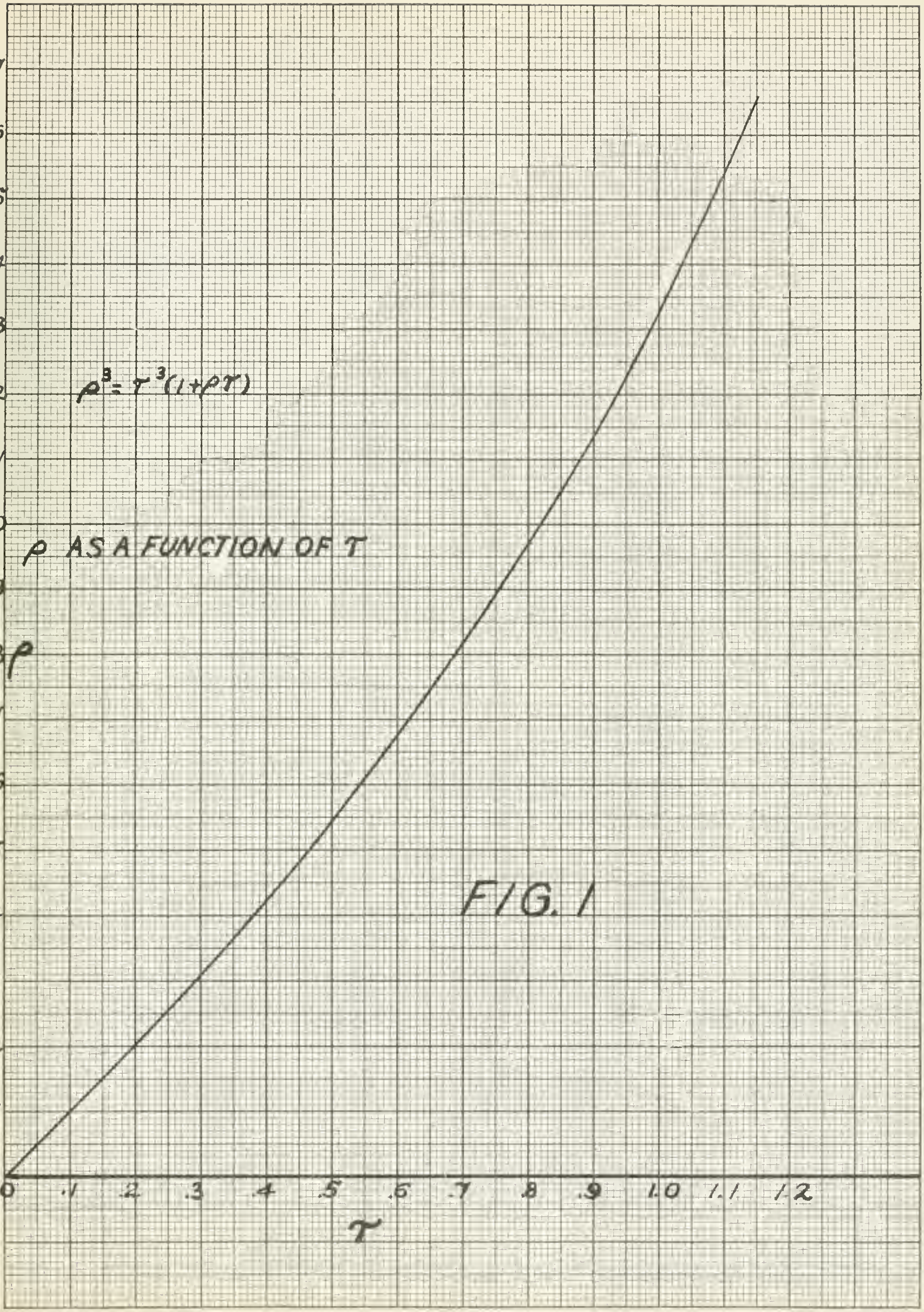
$$\text{Im } y = .0820.$$

This is in agreement with the graph in figure 3 for the values of the parameters $\frac{b}{a} = .8$, $h = 2.4$ and $10^6 \frac{I}{V^{3/2}} = 0.2$. Since d_o is small, the amplification could be computed by the results of this section rather than those of section 3.

Referring back to the graphs in figures 2 and 3, it is seen that the curves for maximum amplification reach a maximum for $h = 2.0$, $\frac{b}{a} = .8$, $h = 2.5$, $\frac{b}{a} = .6$. An explanation of this is afforded by the work of the present section.

It is readily verified that if $\frac{b}{a}$ is fixed and if $0 < \frac{b}{a} < 1$, the function A of Eq. (4.5) has a minimum for $x_o = \dfrac{1}{2(1 - \frac{b}{a})}$. The quantity L of Eq. (4.14) is independent of x_o. Thus it is possible for the roots of Eq. (4.13) to be the same for two different values of x_o. In the work of section 3, the quantity h is close to x_o.

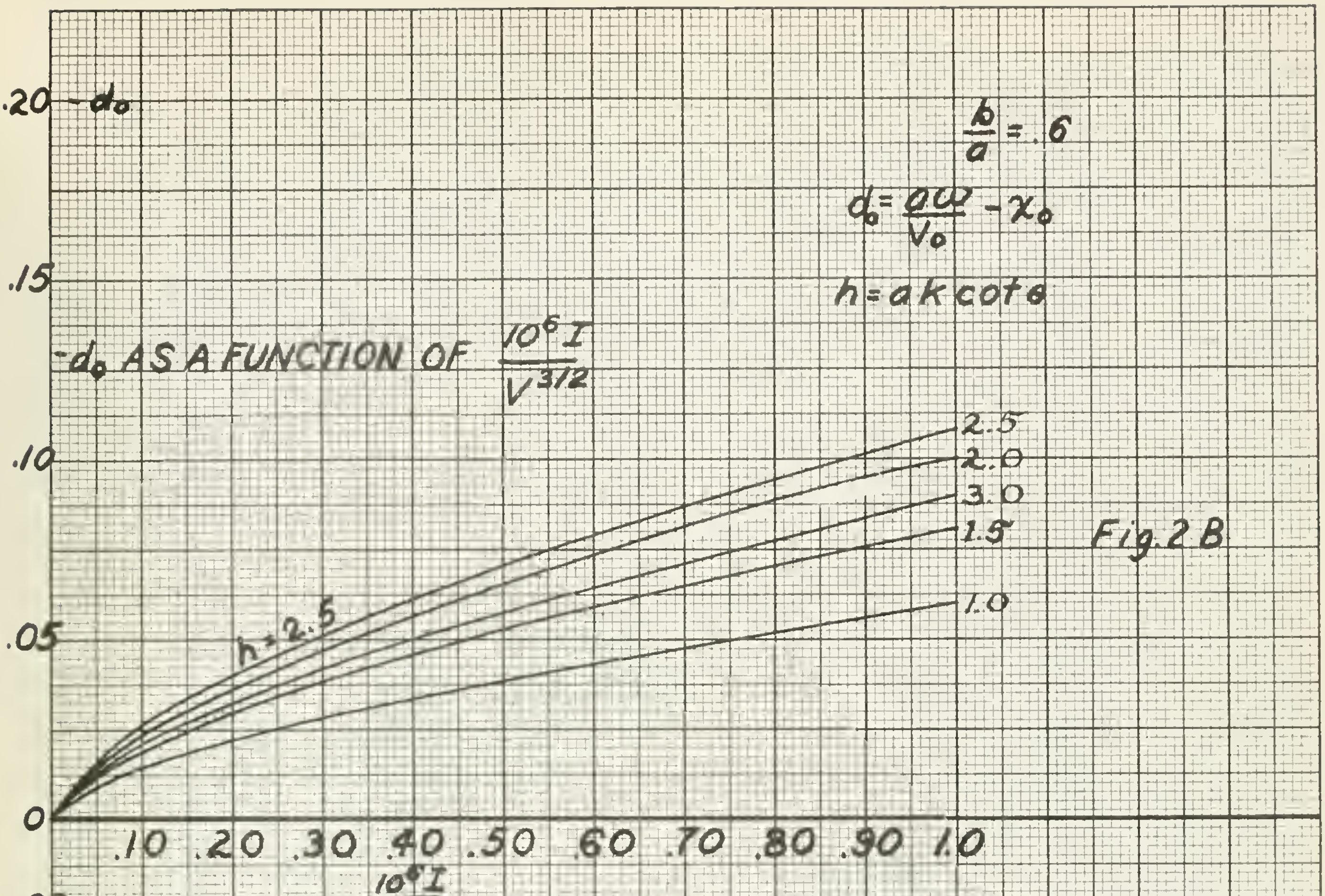
This remark may have practical value for experiment since it indicates that the pitch angle should be so chosen that $ka \cot \theta = 2.0$ for $\frac{b}{a} = .8$ and $ka \cot \Theta = 2.5$ for $\frac{b}{a} = .6$.



$\rho^3 = \tau^3(1+\rho\tau)$

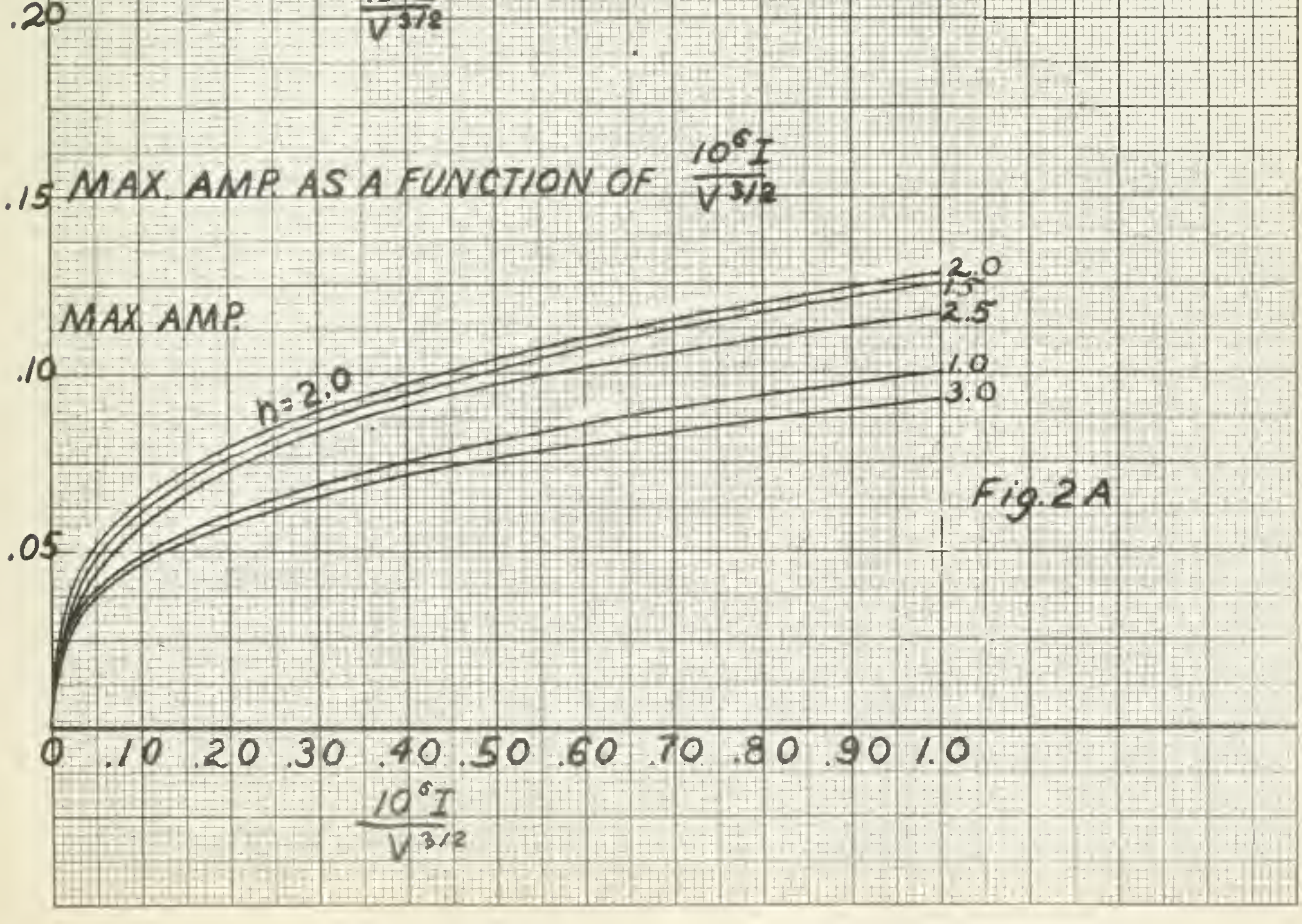
ρ AS A FUNCTION OF τ

FIG. 1



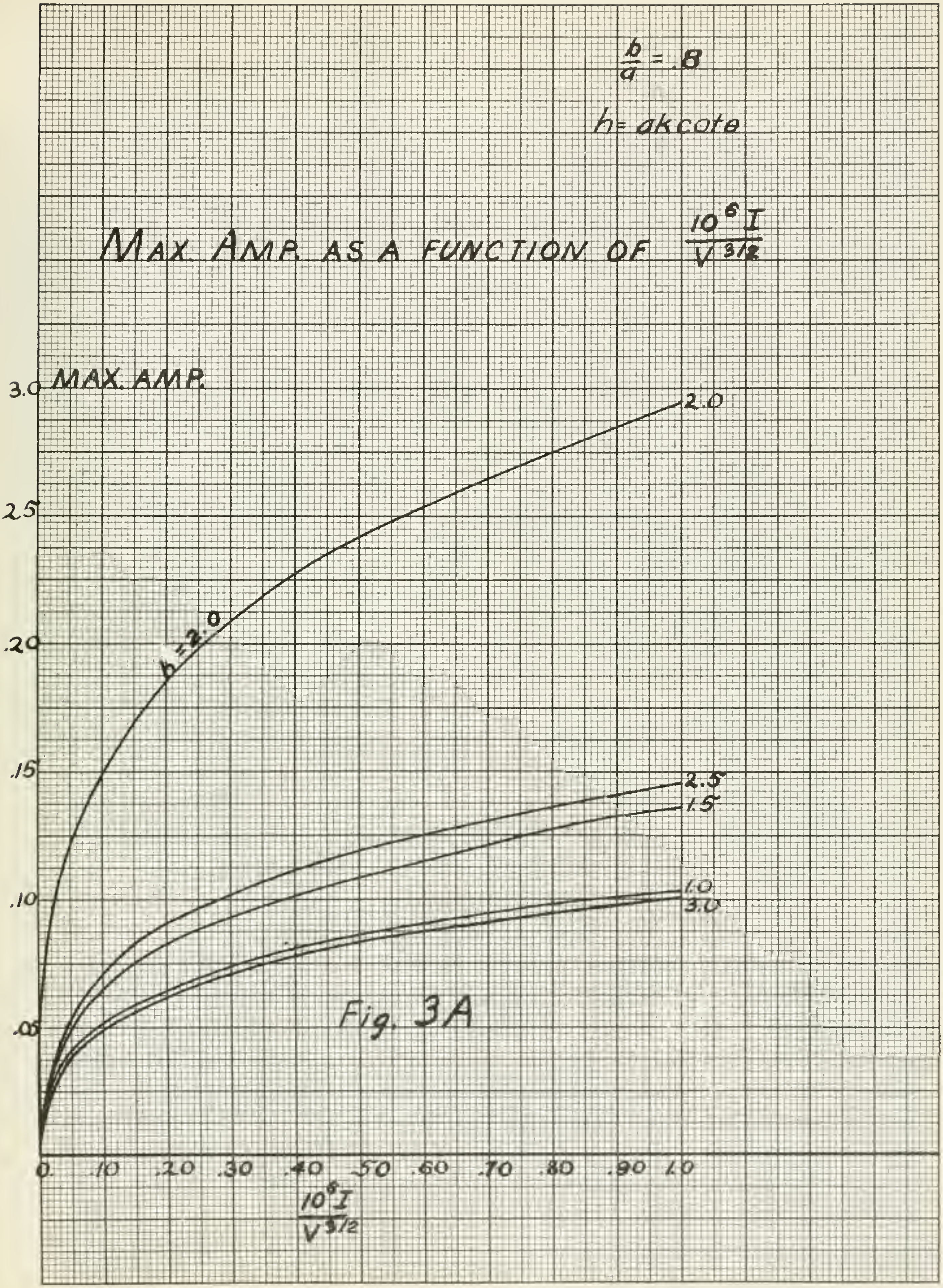
$-d_0$ AS A FUNCTION OF $\frac{10^6 I}{V^{3/2}}$

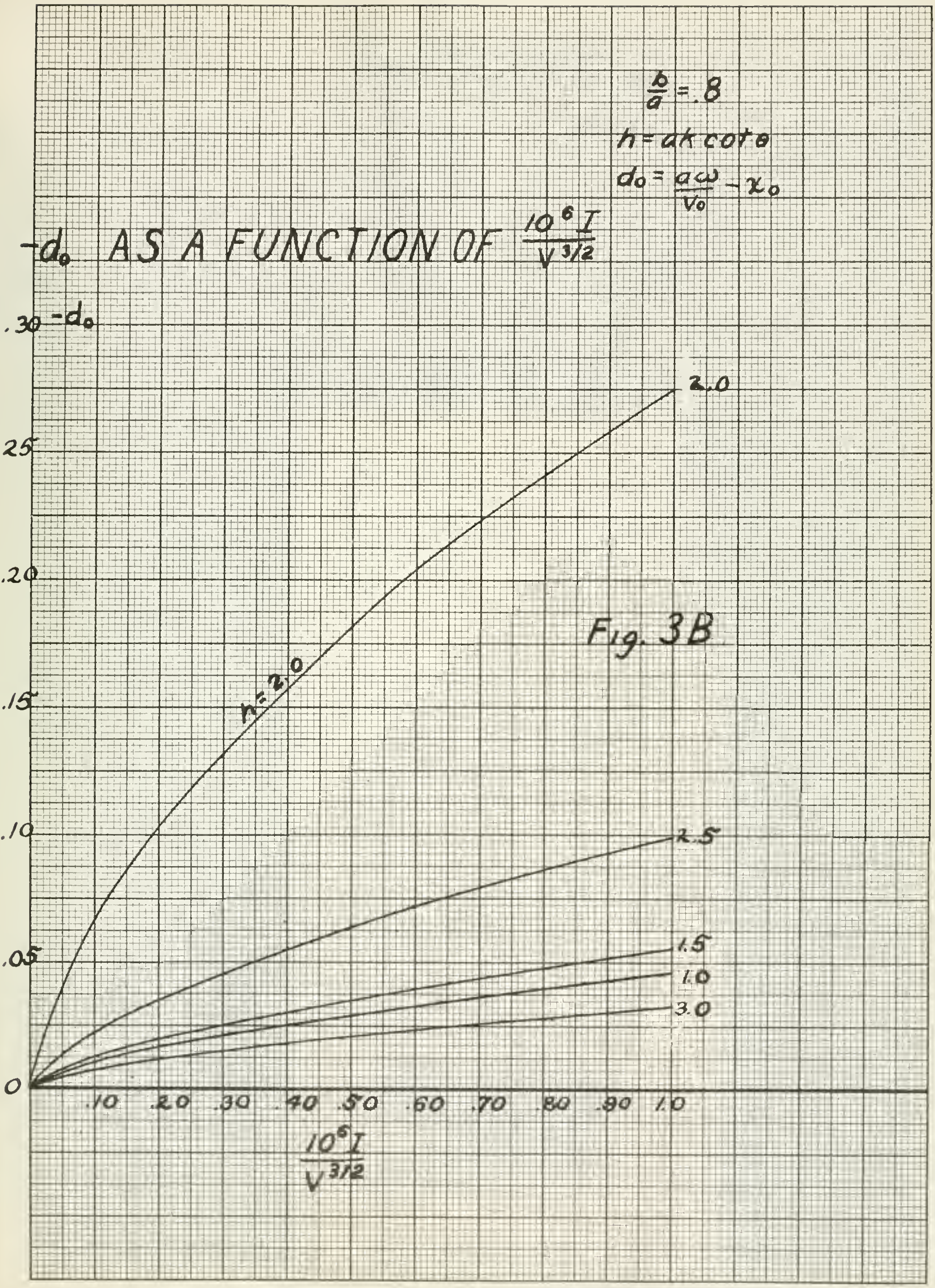
Fig. 2 B



MAX. AMP. AS A FUNCTION OF $\frac{10^6 I}{V^{3/2}}$

Fig. 2 A





$-d_0$ AS A FUNCTION OF $\frac{10^6 I}{V^{3/2}}$

$\frac{b}{a} = .8$

$h = ak \cot \theta$

$d_0 = \frac{a\omega}{v_0} - \chi_0$

Fig. 3B

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